The Hitchin fibration I

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$G = GL_n$

$X/k = \bar{k}$ smooth conn. proj. curve

$$\mathcal{H}iggs_G = \{ (\mathcal{E} \text{ vbdle of rk } n, \phi \in H^0(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X)) \} \ni (\mathcal{E}, \phi)$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\bigoplus_{i=1}^n H^0(X, \Omega^i) \qquad\qquad \ni (a_1, \dots, a_n)$$

with $a_i \in H^0(X, \Omega_X^{\otimes i})$ coefficient of T^{n-i} in characteristic polynomial of ϕ

1. The characteristic polynomial mapping

$k = \bar{k}$ field of char 0

G/k reductive gp, $\mathfrak{g} := \mathrm{Lie}(G) \overset{adj}{\circlearrowleft} G$

$k[\mathfrak{g}] := \mathrm{Sym}^\bullet_k(\mathfrak{g}(k)^\vee)$ so that $\mathfrak{g} = \mathrm{Spec}\, k[\mathfrak{g}]$

Define $\quad \mathfrak{c} := \mathfrak{c}_G := \mathrm{Spec}(k[\mathfrak{g}]^G)$ space of characteristic polynomials

$\chi : \mathfrak{g} \longrightarrow \mathfrak{c}$ induced by $k[\mathfrak{g}]^G \underset{incl}{\hookrightarrow} k[\mathfrak{g}]$, "char. polynomial mapping"

Rem.

- χ factors over $\chi : \underbrace{[\mathfrak{g}/G]}_{\text{stacky quotient}} \longrightarrow \mathfrak{c}$
- G split, $T \subseteq G$ max. torus, $\mathfrak{t} \subseteq \mathfrak{g}$ associated Cartan subalg.

$\rightsquigarrow \quad k[\mathfrak{g}]^G \simeq k[\mathfrak{t}]^W$ for Weyl gp $W = N_G(T)/T$

In particular, $\mathfrak{c} \underset{\substack{\uparrow \\ [\text{NOT } \mathbb{G}_m\text{-equivariant!}]}}{\simeq} \mathbb{A}^r$ for $r := \mathrm{rk}\, G$ (since W generated by reflections)

- $G = Gl_n$, $\mathfrak{g} = \mathfrak{gl}_n = Mat_n(k)$

$\rightsquigarrow$ $k[\mathfrak{g}]^G = k[a_1, \dots, a_n]$ where $\det(X - A) = \sum_{i=0}^{n} a_{n-i} X^i$

$\simeq k[x_1, \dots, x_n]^{S_n} = k[\sigma_1, \dots, \sigma_n]$

where $\sigma_i :=$ ith elem. symm. pol^s

- $G = Sl_n$ or PGl_n

$$\left.\begin{array}{l} \rightsquigarrow\ k[\mathfrak{sl}_n]^{Sl_n} \simeq k[a_1, \dots, a_n]/(a_1) \quad \leftarrow \text{trace zero!} \\ \qquad\qquad\qquad \wr\wr \\ k[\mathfrak{pgl}_n]^{PGl_n} \simeq k[a_2, \dots, a_n] \end{array}\right\} \quad \mathcal{C}_{Sl_n} \simeq \mathcal{C}_{PGl_n}$$

2. Higgs bundles & the Hitchin fibration

X/k sm. proj. conn. curve, $f : X \to \operatorname{Spec} k$ structure morphism

$\mathcal{L} \in \operatorname{Pic}(X)$

$Bun_G := f_*(BG_X)$ where $BG_X :=$ stack of G-torsors "over X"

Recall:
- $f_*(\mathcal{X}) := \mathcal{X}(X \times S)$
- $\mathcal{T} \in Bun_G(S) \rightsquigarrow ad\,\mathcal{T} := \mathcal{T} \overset{G}{\times} \mathfrak{g}$ "adjoint vector bundle"

Def S scheme$/k$

A G-Higgs bdle on X is a pair $(\mathcal{T} \in Bun_G(S), \phi \in H^0(X \times S, ad\,\mathcal{T} \otimes \mathcal{L}))$.

$\mathcal{H}iggs_G := \left(\begin{array}{l}\text{stack of} \\ G\text{-Higgs bdles}\end{array} (\text{wrt } \mathcal{L}) \text{ on } X\right)$.

Prop. V scheme $/S$, H gp scheme $/S$ acting on V over S

S' scheme $/S$

$$\rightsquigarrow [V/H](S') \simeq \{ (\underset{H\text{-torsor}}{\mathcal{T} \to S'},\ \phi \in H^0(S', \mathcal{T} \overset{H}{\underset{S'}{\times}} V)) \}$$

Proof.

$$\begin{tikzcd} \mathcal{T} \arrow[r, "H\text{-equiv.}"'] \arrow[d] & V \arrow[d] \\ S' \arrow[r] & {[V/H]} \end{tikzcd}$$

$$[V/H](S') \simeq \{ (\underset{H\text{-torsor}}{\mathcal{T} \to S'},\ \underset{H\text{ equivar.}/S}{\psi : \mathcal{T} \to V}) \}$$

$$\simeq \{ (\underset{H\text{-torsor}}{\mathcal{T} \to S'},\ \underset{H\text{-equivar.}/\mathcal{T}}{\tilde{\psi} : \mathcal{T} \to \mathcal{T} \times_S V}) \}$$

$$\simeq \{ (\underset{H\text{-torsor}}{\mathcal{T} \to S'},\ \underset{S'\text{- morphism}}{\phi : S' \to \mathcal{T} \overset{H}{\underset{S}{\times}} V}) \} \qquad \square$$

Write $\tilde{\mathfrak{g}} := \mathfrak{g} \times_k X$ $\qquad \tilde{\mathfrak{g}}_{\mathcal{L}} := \tilde{\mathfrak{g}} \otimes L$, $\quad L :=$ geometric bundle associated to $\mathcal{L}$.

$\tilde{G} := G \times_k X$

Prop. $\mathcal{H}iggs_G \simeq f_* ([\tilde{\mathfrak{g}}_{\mathcal{L}} / \tilde{G}])$.

Proof. $\mathcal{H}iggs_G (S) = \{ (\mathcal{T} \to X \times S\ \ \tilde{G}\text{-torsor},$

$\phi \in H^0(X \times S, \mathrm{ad}\, \mathcal{T} \otimes \mathcal{L})) \}$

$$\simeq [\tilde{\mathfrak{g}}_{\mathcal{L}} / \tilde{G}](X \times S) = f_* ([\tilde{\mathfrak{g}}_{\mathcal{L}} / \tilde{G}])(S). \qquad \square$$

Def.
- $\tilde{\mathfrak{c}}_{\mathcal{L}} := \underline{\mathrm{Spec}} \left(\mathrm{Sym}^\bullet (\mathfrak{g}(k)^\vee \otimes_k \mathcal{L}^{-1})^G \right)$ $\quad \left(\simeq \bigoplus_{i=1}^{r} L^{\otimes d_i} \right.$ if $k[\mathfrak{g}]^G = k[a_1, \dots, a_r]$, $d_i := \deg(a_i)$ $\left. \right)$
- $\tilde{\chi}_{\mathcal{L}} : [\tilde{\mathfrak{g}}_{\mathcal{L}} / G] \longrightarrow \tilde{\mathfrak{c}}_{\mathcal{L}}$

- $\mathcal{B} := \mathcal{B}_G := f_*(\tilde{\mathfrak{c}}_{\mathcal{L}})$ Hitchin base

  (≃ scheme associated to the k-v'space $\bigoplus_{i=1}^{r} H^0(X, \mathcal{L}^{\otimes d_i})$
  if $\mathcal{L} \simeq \Omega^1_X$ or $\deg \mathcal{L} \geq 2g_X - 1$)

- $h := h_G := f_*(\tilde{\chi}_{\mathcal{L}}) : \mathcal{H}iggs_G \cong f_*([\tilde{\mathfrak{g}}_{\mathcal{L}}/G]) \longrightarrow \mathcal{B}_G$.

3. Hitchin fibers for Gl_n

$G = Gl_n$, $L = \mathbb{V}(\mathcal{L})$, $p : L \to X$ projection,

$a = (a_1, \dots, a_n) \in \mathcal{B}(k) = \bigoplus_{i=1}^{n} H^0(X, \mathcal{L}^{\otimes i})$

$x \in H^0(L, p^*\mathcal{L})$ tautological section

$\chi_a(x) := \sum_{i=0}^{n} p^*(a_{n-i}) \cdot x^i \in H^0(L, p^*\mathcal{L}^{\otimes n})$ where $a_0 := 1$

Def. $X_a := V(\chi_a(x)) = \{ \eta \in L \mid \chi_a(\eta) = 0 \}$
$= \mathrm{Spec}\left(\mathrm{Sym}^\bullet(\mathcal{L}^{-1})/\mathcal{J}\right)$ for $\mathcal{J} := \mathrm{im}(p^*\mathcal{L}^{-n} \xrightarrow{\chi_a} \mathcal{O}_L)$

$$\begin{tikzcd} \pi : X_a \arrow[r] \arrow[d, phantom, "\cap|"] & X \\ L \arrow[ur] & \end{tikzcd}$$

"spectral cover of X associated with a"

Rem.
- π finite flat of degree n
- a generic $:\Longleftrightarrow$ X_a smooth + integral
  (these a form a nonempty open subset of $\mathcal{B}(k)$)

Thm. For $a \in \mathcal{B}(k)$ sth X_a is integral,
$\exists$ equivalence $h^{-1}(a) \simeq \{ \mathcal{M}$ torsionfree $\mathcal{O}_{X_a}$-modules of rk $1 \}$.

Proof. $\{\mathcal{M}$ torsionfree $\mathcal{O}_{X_a}$-mod. of rk $1\}$

$\underset{\pi \text{ affine}}{\simeq} \{\mathcal{M}'$ torsionfree $\pi_*\mathcal{O}_{X_a}$-mod. of rk $1\}$

$\underset{X_a \text{ integral}}{\simeq} \{\mathcal{E}$ torsionfree $\mathcal{O}_X$-module of rk n with a homom. $\pi_*\mathcal{O}_{X_a} \longrightarrow \mathcal{E}nd(\mathcal{E})\}$

$$\left[\begin{array}{ccc} & & \mathcal{M}' \otimes_{\pi_*\mathcal{O}_{X_a}} k(X_a) \\ \mathcal{M}' & & \wr\wr \\ & & \mathcal{M}' \otimes_{\mathcal{O}_X} k(X) \end{array}\right]$$

$\underset{X \text{ smooth}}{\simeq} \{\mathcal{E}$ v'bdle on X of rk n with homom. $\mathrm{Sym}^\bullet(\mathcal{L}^{-1})/\mathcal{J} \longrightarrow \mathcal{E}nd(\mathcal{E})\}$

$\simeq \{\mathcal{E}$ v'bdle on X of rk n with $\phi : \mathcal{L}^{-1} \longrightarrow \mathcal{E}nd(\mathcal{E})$ s.th. $\chi_a =$ char. polyn. of $\phi\}$

$\simeq h^{-1}(a)$. $\square$

Cor. If $a \in \mathcal{B}(k)$ is generic, then $h^{-1}(a) \simeq \mathcal{P}ic(X_a)$.

$$\begin{array}{ccc} \psi & & \psi \\ \left(\begin{array}{l} \mathcal{E} = \pi_*\mathcal{M}, \\ \phi = \pi_*(x) : \mathcal{E} \to \mathcal{E} \otimes \mathcal{L} \end{array}\right) & \longleftarrow\!\!| & \mathcal{M} \end{array}$$

4. Hitchin fibers for SL_n

$G = SL_n$

Prop. S/k scheme

$\rightsquigarrow \mathcal{H}iggs_{SL_n}(S) \simeq \{(\mathcal{E}$ vbdle on $X \times S$,
$\phi \in H^0(X \times S, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{L})$, (trace-free endom.)
$\alpha : \det(\mathcal{E}) \simeq \mathcal{O}_{X \times S})\}$

Rem. • $\mathcal{B}_{sl_n} = \{ (a_1, \dots, a_n) \in \mathcal{B}_{gl_n} \mid a_1 = 0 \}$

$\rightsquigarrow$ for $a \in \mathcal{B}_{sl_n}(k)$, the X_a and $\pi : X_a \to X$ are defined as in section 3.

Def. $a \in \mathcal{B}_{sl_n}(k)$ generic

• $$\mathrm{Nm} : \mathcal{P}ic(X_a) \longrightarrow \mathcal{P}ic(X)$$
$$\mathcal{M} \longmapsto \det(\pi_* \mathcal{M}) \otimes \det(\pi_* \mathcal{O}_{X_a})^{-1}$$

Fact: morphism of Picard stacks,
in particular $\mathrm{Nm}(\mathcal{M} \otimes \mathcal{N}) \simeq \mathrm{Nm}(\mathcal{M}) \otimes \mathrm{Nm}(\mathcal{N})$.

• $\mathcal{P}rym_{X_a} := \ker(\mathrm{Nm})$ "Prym stack"

Rem. Concretely, $$\mathcal{P}rym_{X_a}(S) = \{ (\mathcal{M} \in \mathcal{P}ic(X_a \times S), \; \alpha : \mathrm{Nm}(\mathcal{M}) \simeq \mathcal{O}_{X \times S}) \}$$

In particular, $\mathrm{Aut}(\mathcal{M}, \alpha) \simeq \mu_n \simeq Z(Sl_n)$.

Prop. $a \in \mathcal{B}_{sl_n}(k)$ generic, $h : \mathcal{H}iggs_{sl_n} \longrightarrow \mathcal{B}_{sl_n}$

$\rightsquigarrow$ $h^{-1}(a) \simeq \mathcal{P}rym_{X_a}$.

Proof. Pick line bdle $\mathcal{M}$ on X_a sth $\det(\pi_* \mathcal{M}) \simeq \mathcal{O}_X$,
i.e. $\mathrm{Nm}(\mathcal{M}) \simeq \det(\pi_* \mathcal{O}_{X_a})^{-1}$.

For $(\mathcal{N}, \alpha) \in \mathcal{P}rym_{X_a}$, get Sl_n - Higgs bdle

$$\Big(\mathcal{E} := \pi_*(\mathcal{M} \otimes \mathcal{N}), \; \phi := \pi_*(x) : \mathcal{E} \to \mathcal{E} \otimes \mathcal{L},$$

$$\det(\mathcal{E}) \simeq \mathrm{Nm}(\mathcal{M} \otimes \mathcal{N}) \otimes \det(\pi_* \mathcal{O}_{X_a}) \simeq \mathrm{Nm}(\mathcal{M}) \otimes \mathrm{Nm}(\mathcal{N}) \otimes \det(\pi_* \mathcal{O}_{X_a}) \simeq \mathcal{O}_X \Big).$$

Conversely, pick $(\mathcal{E}, \phi, \alpha) \in \mathcal{H}iggs_{SL_n}$

$\rightsquigarrow_{Gl_n\text{-case}}$ $(\mathcal{E}, \phi) = \pi_*(\mathcal{M}')$ for some $\mathcal{M}' \in \mathrm{Pic}(X_a)$

$\rightsquigarrow$ $\mathcal{M}' \otimes \mathcal{M}^{-1} \in \mathrm{Prym}\, X_a$. $\square$

5. Hitchin fibers for PGl_n

$\mathcal{X}$ groupoid
$\mathcal{G}$ Picard groupoid
$\alpha : \mathcal{G} \times \mathcal{X} \longrightarrow \mathcal{X}$ faithful action

Define $\mathcal{X}/\mathcal{G}$ as the groupoid with objects the objects of $\mathcal{X}$
and morphisms $\mathrm{Hom}_{\mathcal{X}/\mathcal{G}}(\xi, \eta) := \varinjlim_{g \in \mathcal{G}} \mathrm{Hom}_{\mathcal{X}}(\xi, \alpha(g, \eta))$

Similarly:
$\mathcal{X}$ stack
$\mathcal{G}$ Picard stack acting on $\mathcal{X}$
$\rightsquigarrow$ $\mathcal{X}/\mathcal{G}$:= stackification of prestack $(S \mapsto \mathcal{X}(S)/\mathcal{G}(S))$.

Example. • $\mathrm{Pic}(X)$ acts on Bun_{Gl_n},
with quotient equivalent to Bun_{PGl_n}
(use that $R^2 f_* \mathbb{G}_m = 0$
to get lifting from PGl_n to Gl_n locally)

Set $\mathcal{H}iggs_{Gl_n}^{Tr=0} \subseteq \mathcal{H}iggs_{Gl_n}$ substack of Higgs bdles with trace zero Higgs field.

Prop. $\mathcal{H}iggs_{PGl_n} \simeq \mathcal{H}iggs^{Tr=0}_{Gl_n} / \mathcal{P}ic(X).$

Proof. (1) $\mathcal{B}un_{PGl_n} \simeq \mathcal{B}un_{Gl_n} / \mathcal{P}ic(X)$

(2) Every tracefree Higgsfield on a Gl_n-torsor $\mathcal{T}$ determines uniquely a Higgs field on $\mathcal{T} \times^{Gl_n} PGl_n$. $\square$

Cor. For $a \in \mathcal{B}_{PGl_n}(k) \subseteq \mathcal{B}_{Gl_n}(k)$ generic,

$$
\begin{aligned}
h^{-1}(a) &\simeq \mathcal{P}ic(X_a) / \mathcal{P}ic(X) \\
&\simeq \mathcal{P}rym_{X_a} / \mathcal{P}ic(X)[n] \times \mathbb{Z}/n \\
&\simeq \mathrm{Prym}_{X_a} / \mathrm{Pic}(X)[n] \times \mathbb{Z}/n
\end{aligned}
$$

Proof. 2^{nd} iso comes from

$$
\begin{array}{ccccccccc}
0 & \to & \mathcal{P}ic(X) & \xrightarrow{\pi^*} & \mathcal{P}ic(X_a) & \longrightarrow & K & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathcal{P}ic(X)[n] & \to & \mathcal{P}rym_{X_a} & \longrightarrow & K^0 & \to & 0
\end{array}
$$

$\square$

Nm dual to π^*, so

$$
\begin{array}{ccccccc}
0 \longrightarrow & K^\vee & \to & \mathcal{P}ic(X_a) & \xrightarrow{Nm} & \mathcal{P}ic(X) & \longrightarrow 0 \\
 & \wr\wr & & & & & \\
 & \mathcal{P}rym_{X_a} & & & & &
\end{array}
$$

$$\boxed{\Longrightarrow h^{-1}_{Sl_n}(a) \simeq h^{-1}_{PGl_n}(a)}$$